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The twisted XXZ chain at roots of unity revisited

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Abstract
The twisted XXZ chain alias the six-vertex model is investigated at roots of unity. It is shown that when the twist parameter is chosen to depend on the total spin an infinite-dimensional non-Abelian symmetry algebra can be explicitly constructed in all spin sectors. This symmetry algebra can be identified with the upper or lower Borel subalgebra of the sl_2 loop algebra. The proof uses only the intertwining property of the six-vertex monodromy matrix and the familiar relations of the six-vertex Yang–Baxter algebra.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In recent years there has been renewed interest in the degeneracies exhibited by the integrable six-vertex model and the associated XXZ quantum spin chain,

$$H=\sum_{m=1}^{M}\sigma_m^+\sigma_{m+1}^-+\sigma_m^-\sigma_{m+1}^++\frac{q+q^{-1}}{4}\sigma_m^z\sigma_{m+1}^z \qquad \sigma_{M+1}^{\pm}\equiv\sigma_1^{\pm} \qquad \sigma_{M+1}^z\equiv\sigma_1^z \tag{1}$$

when the anisotropy parameter is evaluated at roots of unity $q^N=1$. In [1], Deguchi *et al* showed for the commensurate sectors $2S^z=0 \bmod N$ (with S^z being the total spin) that the model with periodic boundary conditions exhibits an $\widetilde{sl}_2=sl_2\otimes\mathbb{C}[t,t^{-1}]$ loop algebra symmetry. Outside these commensurate sectors the algebraic structure of the symmetry algebra has so far not been established except for $N=4$, i.e. the case of vanishing anisotropy parameter, the XX model and a numerical construction for $N=3$, see [1] for details.

In two recent works [2, 3] the twisted XXZ chain at roots of unity has been investigated,

$$H^{\lambda}=\sum_{m=1}^{M}\sigma_m^+\sigma_{m+1}^-+\sigma_m^-\sigma_{m+1}^++\frac{q+q^{-1}}{4}\sigma_m^z\sigma_{m+1}^z \qquad \sigma_{M+1}^{\pm}\equiv\lambda^{\pm1}\sigma_1^{\pm} \qquad \sigma_{M+1}^z\equiv\sigma_1^z. \tag{2}$$

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In [2] various operators have been constructed which (anti)commute with the twisted XXZ Hamiltonian and the associated transfer matrix. Except for the cases $\lambda = -1$ the algebraic structure underlying these operators has not been identified. Similarly as for the periodic case the $\widetilde{sl}_2$ symmetry algebra for $\lambda = -1$ has been restricted to certain commensurate spin sectors. The discussion in [2] has also been extended to include the case of the inhomogeneous chain.

In the second work [3], the construction of operators creating complete strings for the periodic homogeneous chain carried out in [4] has been generalized to cover also the twisted and inhomogeneous case. The construction of the symmetry algebra underlying the degeneracies in the spectra of the Hamiltonian and the transfer matrix has not been investigated.

In this paper it is shown that when the twist parameter is chosen to depend on the total spin, i.e. $\lambda = q^{\pm 2S^z}$, the quantum spin-chain Hamiltonian and the associated twisted six-vertex transfer matrix exhibit infinite-dimensional non-Abelian symmetries and their algebraic structure is identified with the lower, respectively upper, Borel subalgebra $U(b_\mp) \subset U(\widetilde{sl}_2)$. In contrast to the case of periodic boundary conditions the construction of the symmetry algebra is given explicitly for *all* spin sectors at arbitrary roots of unity. In the spin sectors $2S^z = 0 \bmod N$ one obviously recovers the periodic chain and the symmetry is enhanced to the full loop algebra $U(\widetilde{sl}_2)$ reproducing the aforementioned result of [1]. However, also for the periodic case we give a novel proof of the symmetry which only uses the framework of the algebraic Bethe ansatz [5] and quantum group theory [6, 7]. In particular, it avoids having first to prove translation invariance, cf [1, 8, 9]. The extension to the inhomogeneous case is also discussed.

2. The twisted six-vertex model

The starting point of our discussion is the six-vertex R-matrix which is given by

$$R(z,q) = \tfrac{a+b}{2} 1 \otimes 1 + \tfrac{a-b}{2} \sigma^z \otimes \sigma^z + c\sigma^+ \otimes \sigma^- + c'\sigma^- \otimes \sigma^+ \tag{3}$$

where we choose the following parametrization of the Boltzmann weights:

$$a = 1 \qquad b = \frac{1-z}{1-zq^2} q \qquad c = \frac{1-q^2}{1-zq^2} \qquad c' = cz. \tag{4}$$

Here z denotes the (multiplicative) spectral parameter and q is the deformation parameter appearing in the spin-chain Hamiltonians (1) and (2). Central to our discussion will be the properties of the (inhomogeneous) six-vertex monodromy matrix which one usually decomposes over the two-dimensional auxiliary space,

$$R_{0M}(z/\zeta_M) \cdots R_{01}(z/\zeta_1) = \sigma^+\sigma^- \otimes A + \sigma^+ \otimes B + \sigma^- \otimes C + \sigma^-\sigma^+ \otimes D. \tag{5}$$

The explicit dependence on the spectral parameter and the inhomogeneity parameters $\zeta = (\zeta_1, \dots, \zeta_M)$ will be often suppressed in the notation in order to unburden the formulae. The twisted six-vertex transfer matrix is defined as the trace

$$T^\lambda(z) = \operatorname*{Tr}_0 \lambda^{\frac{\sigma^z \otimes \mathbf{1}}{2}} R_{0M}(z/\zeta_M) \cdots R_{01}(z/\zeta_1) = \lambda^{\frac{1}{2}} A(z) + \lambda^{-\frac{1}{2}} D(z). \tag{6}$$

For the homogeneous chain $\zeta_1 = \cdots = \zeta_M = 1$ we obtain up to an additive constant the spin-chain Hamiltonian (2) as the following logarithmic derivative:

$$H^\lambda = (q - q^{-1}) T^\lambda(z)^{-1} z \frac{\mathrm{d}}{\mathrm{d}z} T^\lambda(z) \Big|_{z=1} + M \frac{q + q^{-1}}{2}. \tag{7}$$

Obviously, the twist does not alter the algebraic relations of the Yang–Baxter algebra defined in terms of $\{A, B, C, D\}$ in (5). In order to discuss the symmetries of (2) and (6) when the deformation parameter q is a root of unity we first establish a number of relations between the Chevalley–Serre basis of the quantum group $U_q(\widetilde{sl}_2)$ and the matrix elements of the monodromy matrix (5) for generic q and λ.

3. The Chevalley–Serre basis of $U_q(\widetilde{sl}_2)$

It is well known that the underlying algebraic structure of the six-vertex model is the quantum loop algebra $U_q(\widetilde{sl}_2)$. Its algebraic definition [6, 7] in terms of the Chevalley–Serre basis is

$$k_i\, \mathrm{e}_j k_i^{-1} = q^{\mathcal{A}_{ij}}\, \mathrm{e}_j \qquad k_i f_j k_i^{-1} = q^{-\mathcal{A}_{ij}} f_j \qquad k_i k_j = k_j k_i \qquad i, j = 0, 1 \tag{8}$$

where the Cartan matrix reads

$$\mathcal{A} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

In addition one has to impose for $i \neq j$ the Chevalley–Serre relations

$$\begin{aligned} &\mathrm{e}_i^3\, \mathrm{e}_j - [3]_q\, \mathrm{e}_i^2\, \mathrm{e}_j\, \mathrm{e}_i + [3]_q\, \mathrm{e}_i\, \mathrm{e}_j\, \mathrm{e}_i^2 - \mathrm{e}_j\, \mathrm{e}_i^3 = 0 \\ &f_i^3 f_j - [3]_q f_i^2 f_j f_i + [3]_q f_i f_j f_i^2 - f_j f_i^3 = 0. \end{aligned} \tag{9}$$

The quantum algebra can be made into a Hopf algebra upon defining a coproduct which we choose to be

$$\begin{aligned} &\Delta(e_i) = 1 \otimes e_i + k_i \otimes e_i \qquad &&\Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i \\ &\Delta(k_i) = k_i \otimes k_i &&i = 0, 1. \end{aligned} \tag{10}$$

The opposite coproduct Δ^{op} is obtained by permuting the two factors. The six-vertex R-matrix intertwines the two coproduct structures in the case of the spin $1/2$ representation, i.e.

$$R(z/\zeta)(\pi_z \otimes \pi_\zeta)\Delta(x) = (\pi_z \otimes \pi_\zeta)\Delta^{\mathrm{op}}(x) R(z/\zeta) \tag{11}$$

with the representation $\pi_z : U_q(\widetilde{sl}_2) \to \mathrm{End}\,\mathbb{C}^2$ given in terms of Pauli matrices by

$$\begin{aligned} &\pi_z(e_0) = z\sigma^- \qquad &&\pi_z(f_0) = z^{-1}\sigma^+ \qquad &&\pi_z(k_0) = q^{-\sigma^z} \\ &\pi_z(e_1) = \sigma^+ &&\pi_z(f_1) = \sigma^- &&\pi_z(k_1) = q^{\sigma^z}. \end{aligned} \tag{12}$$

From the fusion relation $(1 \otimes \Delta)R = R_{13}R_{12}$ an analogous intertwining relation follows for the monodromy matrix (5) with regard to the quantum group generators on the quantum spin chain $\pi_{\zeta_1} \otimes \cdots \otimes \pi_{\zeta_M}$,

$$\begin{aligned} K_i &= q^{\varepsilon_i \sigma^z} \otimes \cdots \otimes q^{\varepsilon_i \sigma^z} = q^{\varepsilon_i 2S^z} \\ E_i &= \sum_{m=1}^{M} \zeta_m^{\delta_{i0}} q^{\varepsilon_i \sigma^z} \otimes \cdots q^{\varepsilon_i \sigma^z} \otimes \sigma^{\varepsilon_i}_{m^{\mathrm{th}}} \otimes 1 \cdots \otimes 1 \\ F_i &= \sum_{m=1}^{M} \zeta_m^{-\delta_{i0}} 1 \otimes \cdots 1 \otimes \sigma^{-\varepsilon_i}_{m^{\mathrm{th}}} \otimes q^{-\varepsilon_i \sigma^z} \cdots \otimes q^{-\varepsilon_i \sigma^z} \qquad \varepsilon_i := (-1)^{i+1}. \end{aligned} \tag{13}$$

Here $i = 0, 1$ as before[1]. From the intertwining property of the monodromy matrix one then obtains the commutators

$$[A, K_1] = [D, K_1] = 0 \qquad K_1 B K_1^{-1} = q^{-2} B \qquad K_1 C K_1^{-1} = q^2 C \tag{14}$$

[1] Note that we have chosen to work in the homogeneous gradation in (12) in accordance with the parametrization (4) of the Boltzmann weights. Equally well, we could have used the principal gradation in which the six-vertex R-matrix (3) is symmetric. Then all Chevalley–Serre generators in (12) would acquire a spectral parameter dependence and the generators (13) would correspond to those discussed in equation (50) of [2]. The choice of the gradation does not alter the algebraic structure.

and

$$\begin{array}{lll} [E_1, A]_q = -K_1 C & [E_1, B]_{q^{-1}} = A - K_1 D & [E_1, C]_q = 0 \\ [E_1, D]_{q^{-1}} = C & [A, F_1]_{q^{-1}} = -B K_1^{-1} & [B, F_1]_{q^{-1}} = 0 \\ [C, F_1]_q = A - D K_1^{-1} & [D, F_1]_q = B. & \end{array} \tag{15}$$

Here $[x, y]_q = xy - qyx$. The commutation relations for the affine generators $\{E_0, F_0, K_0\}$ are obtained by the simultaneous replacement

$$(A, B, C, D) \to (D, z^{-1}C, zB, A) \qquad \text{and} \qquad (E_1, F_1, K_1) \to (E_0, F_0, K_0). \tag{16}$$

Note that for the homogeneous case $\zeta_1 = \cdots = \zeta_M = 1$ the above algebra automorphism is implemented by the spin-reversal operator $\mathfrak{R} = \sigma^x \otimes \cdots \otimes \sigma^x$,

$$\mathfrak{R} E_i \mathfrak{R} = E_{i+1} \qquad \mathfrak{R} F_i \mathfrak{R} = F_{i+1} \qquad \mathfrak{R} K_i \mathfrak{R} = K_{i+1} \qquad i \in \mathbb{Z}_2. \tag{17}$$

Instead of the spin 1/2 representation (12) one might equally well use evaluation representations of higher spin in the definition of the spin-chain generators (13), as has similarly been done in [3]. As long as the auxiliary space is not altered the form of the commutation relations (14) and (15) is unchanged. From (15) one now deduces by a straightforward computation the following relations for the twisted six-vertex transfer matrix[2]:

$$\begin{aligned} E_1^n T^\lambda &= \left(q^n \lambda^{\frac{1}{2}} A + q^{-n} \lambda^{-\frac{1}{2}} D\right) E_1^n + \lambda^{-\frac{1}{2}} [n]_q (1 - \lambda K_1) C E_1^{n-1} \\ E_0^n T^\lambda &= \left(q^{-n} \lambda^{\frac{1}{2}} A + q^n \lambda^{-\frac{1}{2}} D\right) E_0^n + \lambda^{\frac{1}{2}} z [n]_q (1 - \lambda^{-1} K_0) B E_0^{n-1} \end{aligned} \tag{18}$$

and

$$\begin{aligned} F_1^n T^\lambda &= \left(q^n \lambda^{\frac{1}{2}} A + q^{-n} \lambda^{-\frac{1}{2}} D\right) F_1^n - \lambda^{-\frac{1}{2}} [n]_q q^{-n} \left(1 - \lambda K_1^{-1}\right) F_1^{n-1} B \\ F_0^n T^\lambda &= \left(q^{-n} \lambda^{\frac{1}{2}} A + q^n \lambda^{-\frac{1}{2}} D\right) F_0^n - \lambda^{\frac{1}{2}} z^{-1} [n]_q q^{-n} \left(1 - \lambda^{-1} K_0^{-1}\right) F_0^{n-1} C. \end{aligned} \tag{19}$$

We are now in a position to discuss the symmetry algebras of the twisted six-vertex transfer matrix at roots of unity.

4. Infinite non-Abelian symmetries at $q^N = 1$

Henceforth we set the deformation parameter q to be a primitive root of unity of order $N \geqslant 3$. This entails significant changes in the algebraic structure of the quantum loop algebra $U_q(\widetilde{sl}_2)$. There now exist two versions of the algebra, one of them, which we denote by $U_q(\widetilde{sl}_2)$, has an enlarged centre compared to generic q. Its representation theory has been discussed to some extent in [10]. The other version from which we will obtain the symmetry generators is the restricted quantum algebra $U_q^{\text{res}}(\widetilde{sl}_2)$. It can be realized as automorphisms over $U_q(\widetilde{sl}_2)$. Details on its representation theory can be found in [11]. For the present purposes it will be important that for evaluation representations of the form (12) used in the definition of the quantum spin chain one can write down explicit formulae for the generators of $U_q^{\text{res}}(\widetilde{sl}_2)$: for some $\tilde{q}$ with $\tilde{q}^N \neq 1$ and $n \in \mathbb{N}$ we set

$$\begin{aligned} E_1^{(n)} = \lim_{\tilde{q} \to q} E_1^n(\tilde{q})/[n]_{\tilde{q}}! = \sum_{m_i} q^{n\sigma^z} \otimes \cdots \otimes \sigma^+_{m_1^{\text{th}}} \otimes q^{(n-1)\sigma^z} \cdots \otimes \sigma^+_{m_2^{\text{th}}} \\ \otimes q^{(n-2)\sigma^z} \cdots q^{\sigma^z} \otimes \sigma^+_{m_n^{\text{th}}} \otimes 1 \cdots \otimes 1 \end{aligned}$$

[2] Note that these equations imply similar commutation relations for the twisted XXZ Hamiltonian (2) when differentiating w.r.t. the spectral parameter. The result differs from one obtained in (2.60) of [12] which contains a typo.

$$E_0^{(n)} = \lim_{\tilde q\to q} E_0^n(\tilde q)/[n]_{\tilde q}! = \sum_{m_i} \zeta_{m_1}\cdots\zeta_{m_n} q^{-n\sigma^z}\otimes\cdots\otimes\sigma^-_{m_1^{\text{th}}}\otimes q^{(1-n)\sigma^z}\cdots\otimes\sigma^-_{m_2^{\text{th}}}$$
$$\otimes q^{(2-n)\sigma^z}\cdots q^{-\sigma^z}\otimes\sigma^-_{m_n^{\text{th}}}\otimes 1\cdots\otimes 1$$

and

$$F_1^{(n)} = \lim_{\tilde q\to q} F_1^n(\tilde q)/[n]_{\tilde q}! = \sum_{m_i} 1\otimes\cdots 1\otimes\sigma^-_{m_1^{\text{th}}}\otimes q^{-\sigma^z}\cdots\otimes\sigma^-_{m_2^{\text{th}}}$$
$$\otimes q^{-2\sigma^z}\cdots q^{-(n-1)\sigma^z}\otimes\sigma^-_{m_n^{\text{th}}}\otimes\cdots\otimes q^{-n\sigma^z}$$

$$F_0^{(n)} = \lim_{\tilde q\to q} F_0^n(\tilde q)/[n]_{\tilde q}! = \sum_{m_i} \zeta_{m_1}^{-1}\cdots\zeta_{m_n}^{-1} 1\otimes\cdots 1\otimes\sigma^+_{m_1^{\text{th}}}\otimes q^{\sigma^z}\cdots\otimes\sigma^+_{m_2^{\text{th}}}$$
$$\otimes q^{2\sigma^z}\cdots q^{(n-1)\sigma^z}\otimes\sigma^+_{m_n^{\text{th}}}\otimes\cdots\otimes q^{n\sigma^z}.$$

Here the sums are restricted to the n-tuples $(m_1,\ldots,m_n)$ with $1\leqslant m_1<\cdots<m_n\leqslant M$. Remarkably, the subalgebra generated by the above operators with powers equal to

$$n = N' := \begin{cases} N & N\text{ odd} \\ N/2 & N\text{ even}\end{cases}$$

is isomorphic to the 'classical' loop algebra $U(\widetilde{sl}_2)$ [1, 8, 11]. The projection $U_q^{\text{res}}(\widetilde{sl}_2)\to U(\widetilde{sl}_2)$ is referred to as the quantum Frobenius homomorphism [11]. In order to stress that this is an infinite-dimensional algebra we rewrite $U(\widetilde{sl}_2)$ in terms of its mode basis

$$\begin{aligned} h_{m+n} &= \left[x_m^+, x_n^-\right] & \left[h_m, x_n^\pm\right] &= \pm 2x^\pm_{m+n} \\ [h_m, h_n] &= 0 & \left[x^\pm_{m+1}, x_n^\pm\right] &= \left[x_m^\pm, x^\pm_{n+1}\right]. \end{aligned} \tag{20}$$

The generators $\left\{x_m^\pm, h_m\right\}_{m\in\mathbb{Z}}$ can be successively obtained from the Chevalley–Serre basis via the correspondence [11]

$$E_1^{(N')}\to x_0^+ \qquad F_1^{(N')}\to x_0^- \qquad E_0^{(N')}\to x_1^- \qquad F_0^{(N')}\to x_{-1}^+ \qquad 2S^z/N'\to h_0. \tag{21}$$

For later purposes let us identify the upper and lower Borel subalgebras $U(b_\pm)\subset U(\widetilde{sl}_2)$. In terms of the Chevalley–Serre basis they are generated by $\left\{E_0^{(N')}, E_1^{(N')}, 2S^z/N'\right\}$ and $\left\{F_0^{(N')}, F_1^{(N')}, 2S^z/N'\right\}$, respectively. In the mode basis they simply correspond to the algebras associated with the positive and negative integers,

$$U(b_+) = \left\{x_m^\pm, h_m\right\}_{m\in\mathbb{Z}_{>0}}\cup\left\{x_0^+, h_0\right\} \qquad\text{and}\qquad U(b_-) = \left\{x_m^\pm, h_m\right\}_{m\in\mathbb{Z}_{<0}}\cup\left\{x_0^-, h_0\right\}. \tag{22}$$

We are now in a position to discuss the various symmetries of the twisted six-vertex model at roots of unity. Taking the root-of-unity limit in (18) and (19) we obtain the relations

$$\begin{aligned} E_1^{(N')}T^\lambda &= q^{N'}T^\lambda E_1^{(N')} + \lambda^{-\frac12}(1-\lambda K_1)CE_1^{(N'-1)} \\ E_0^{(N')}T^\lambda &= q^{N'}T^\lambda E_0^{(N')} + \lambda^{\frac12}z(1-\lambda^{-1}K_0)BE_0^{(N'-1)} \end{aligned} \tag{23}$$

and

$$\begin{aligned} F_1^{(N')}T^\lambda &= F_1^{(N')}q^{N'}T^\lambda - \lambda^{-\frac12}q^{-N'}\left(1-\lambda K_1^{-1}\right)F_1^{(N'-1)}B \\ F_0^{(N')}T^\lambda &= F_0^{(N')}q^{N'}T^\lambda - \lambda^{\frac12}z^{-1}q^{-N'}\left(1-\lambda^{-1}K_0^{-1}\right)F_0^{(N'-1)}C. \end{aligned} \tag{24}$$

Table 1. The various symmetry algebras for the twisted six-vertex model at a primitive root of unity $q^N = 1$ and the spin sectors in which they have been constructed explicitly.

Twist	Symmetry	Spin sector
$\lambda = 1$	$U(\widetilde{sl}_2)$	$2S^z = 0 \bmod N$
$\lambda = q^n$	$U(b_\pm)$	$2S^z = \mp n \bmod N$
$\lambda = q^{\mp 2S^z}$	$U(b_\pm)$	All sectors

Thus, upon inserting $K_1 = K_0^{-1} = q^{2S^z}$ we now infer immediately that whenever the terms in brackets vanish we obtain a symmetry algebra. For periodic boundary conditions, $\lambda = 1$, we recover the previously obtained loop algebra symmetry $U(\widetilde{sl}_2)$ in the commensurate sectors $2S^z = 0 \bmod N$ [1]. For twisted boundary conditions with $\lambda = q^{\mp n}, 0 < n < N$, we apparently only obtain 'half' the symmetry algebra, namely $U(b_\pm)$, in the spin sectors $2S^z = \pm n \bmod N$. This is due to the fact that the Cartan generators K_i appear with inverse powers in (24) compared to those in (23). Obviously, we again recover the full loop algebra as a symmetry for even roots of unity and $n = N'$, i.e. the case of antiperiodic boundary conditions $\lambda = q^{N'} = -1$ discussed in [2].

So far all discussed symmetries have only been established for certain commensurate spin sectors. If we choose, however, the twist parameter to depend on the total spin the infinite-dimensional non-Abelian algebras (22) extend to a symmetry for *all* spin sectors. Namely, we now consider the transfer matrices

$$T^\pm(z) = \operatorname*{Tr}_0 q^{\pm\sigma^z \otimes S^z} R_{0M}(z/\zeta_M) \cdots R_{01}(z/\zeta_1) = q^{\pm S^z} A(z) + q^{\mp S^z} D(z). \quad (25)$$

At first sight one might be worried that the twist parameter is now an operator instead of a mere constant. But according to (14) we have $\left[A, q^{S^z}\right] = \left[D, q^{S^z}\right] = 0$ whence upon employing the standard relations of the six-vertex Yang–Baxter algebra the integrability of the model is ensured, i.e.

$$[T^\pm(z), T^\pm(w)] = [A(z), D(w)] + [D(z), A(w)] = 0. \quad (26)$$

Thus, all results generalize in a straightforward manner to this case. The only difference is that in the commutation relations (18) and (19) we now collect additional factors $\tilde{q}^{\pm n}$ on the left-hand side of the equations as we have to 'pull' $\tilde{q}^{\pm S^z}$ past the generators E_i^n, F_i^n,

$$\begin{aligned} E_1^n\left(\tilde{q}^{-n-S^z} A + \tilde{q}^{n+S^z} D\right) &= \left(\tilde{q}^{n-S^z} A + \tilde{q}^{-n+S^z} D\right) E_1^n \\ F_1^n\left(\tilde{q}^{-n+S^z} A + \tilde{q}^{n-S^z} D\right) &= \left(\tilde{q}^{n+S^z} A + \tilde{q}^{-n-S^z} D\right) F_1^n. \end{aligned} \quad (27)$$

The relations for the affine step operators follow from (16). As a consequence the transfer matrices $T^\pm$ now always commute with the generators of (22) in the root-of-unity limit $\tilde{q} \to q$ (instead of anticommuting for even roots of unity cf equations (23) and (24)),

$$[T^+(z), U(b_-)] = 0 \qquad \text{and} \qquad [T^-(z), U(b_+)] = 0. \quad (28)$$

These symmetries hold for all spin sectors and are the main result of this paper. Note that the case of periodic boundary conditions [1] is contained in these models for the sectors $2S^z = 0 \bmod N$, where both transfer matrices coincide and the symmetry is enhanced to the full loop algebra.

5. Conclusions

Let us summarize the established symmetry algebras for the twisted inhomogeneous six-vertex model and their corresponding commensurate sectors in table 1. Note that the above findings

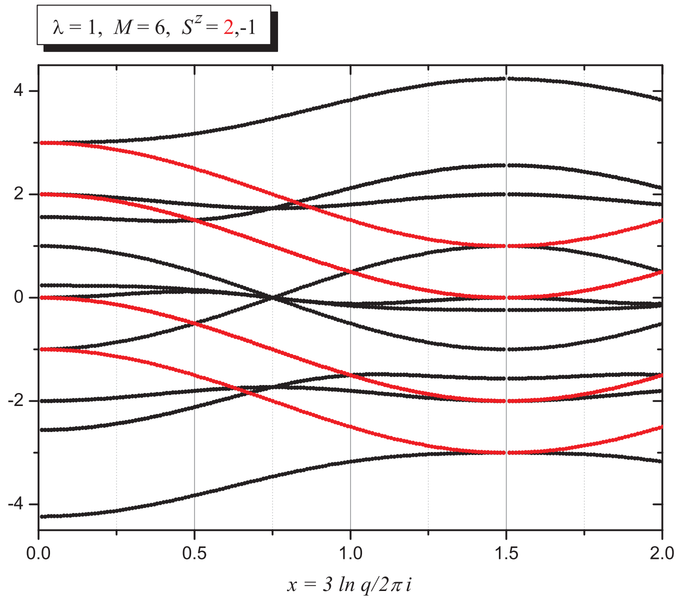

Figure 1. The eigenvalues for the periodic XXZ Hamiltonian (1) with $M = 6$ as a function of the deformation parameter $q = \exp 2\pi \mathrm{i} x/3$. The eigenvalues for the spin sector $S^z = 2$ are shown in red colour. The eigenvalues corresponding to the two inner lines are each doubly degenerate. The eigenvalues corresponding to the spin $S^z = -1$ sector are shown in black colour, also here some of them are doubly degenerate. At the root-of-unity points $x = 1/2, 1$ we see that additional degeneracies occur between eigenvalues from the two different spin sectors. Note that these are incommensurate sectors. The distinguished points $x = 3/4, 3/2$ correspond to the XX model and the case when $q = -1$.

do not exclude the possibility that the symmetries found [1, 2] for the boundary conditions $\lambda \neq q^{\pm 2S^z}$ can be extended to all spin sectors as well. For periodic boundary conditions $\lambda = 1$ it has been argued in [1] that one might have to use projection operators to obtain the symmetry algebra in the incommensurate sectors. As mentioned in the introduction this has been explicitly demonstrated at the free fermion point, i.e. the XX model ($N/2 = N' = 2$). For $N' \geqslant 2$ it has been proved that operators of the type $E_1^n E_0^n$, $E_1^{N'-n} E_0^{N'-n}$, etc commute with the transfer matrix when $\lambda = 1$ and $2S^z = 2n \bmod N$, cf equation (3.42) and section 3.5, appendix A.5 in [1]. In equation (3.43) of the same work eight operators are stated which should (anti)commute with the transfer matrix in the incommensurate sector $2S^z = 2n \bmod N$ and a numerical procedure is described how the loop algebra relations have been verified for $N' = 3$.

For the transfer matrices (25) we obviously do not need any projection operators to extend the symmetry to all spin sectors, which indicates that these models possess a higher level of degeneracies in their spectrum compared to the other boundary conditions. That this is indeed the case has been numerically verified in the spin sectors $S^z = 2, -1$ of the $M = 6$ spin chain when $q^3 = 1$; see figures 1 and 2. Furthermore, our results for the twisted case when $\lambda \neq q^{\pm 2S^z}$ suggest that in the incommensurate sectors one might also encounter a smaller symmetry algebra as the spin sector $2S^z = 0 \bmod N$ is clearly distinguished.

We emphasize again that in comparison with previously established non-Abelian symmetries, e.g. the *finite* quantum group symmetry $U_q(sl_2)$ for the chain with open boundary conditions [12], the symmetries established here involve *infinite*-dimensional algebras which

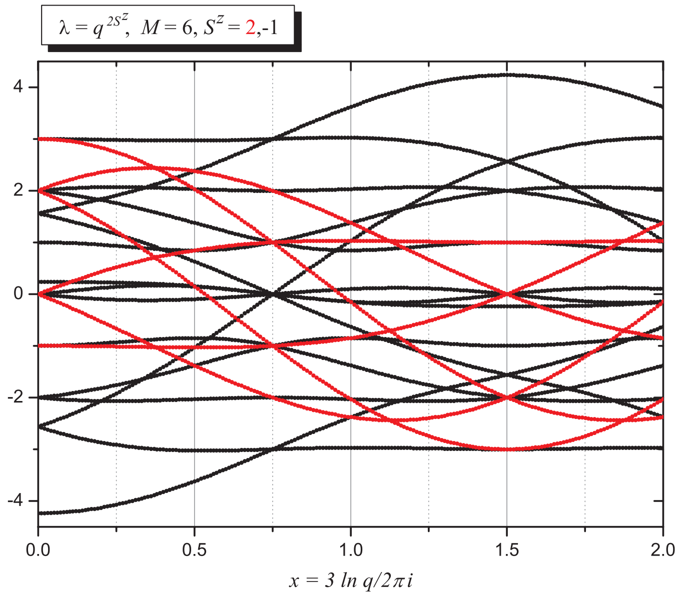

Figure 2. The eigenvalues for the twisted XXZ Hamiltonian (2) with the twist depending on the spin $\lambda = q^{2S^z}$. As in the periodic case the eigenvalues in the spin sector $S^z = 2$ and $S^z = -1$ are displayed in red and black, respectively. Unlike in the periodic case the degeneracies of the Hamiltonian within the respective spin sectors are lifted. In addition, we see that at the root-of-unity values $x = 1/2, 1$ all six eigenvalues of the $S^z = 2$ sector become degenerate with eigenvalues in the $S^z = -1$ sector. These degeneracies indicate the discussed $U(b_\pm)$ symmetries.

impose more powerful restrictions. The next step in this context is to relate the representation theory of these algebras to the Bethe ansatz.

For periodic boundary conditions $\lambda = 1$ this has already partially been done in [4, 14–16]. In [4] creation operators involving complete strings have been constructed which involve two polynomials depending on the Bethe roots. Based on numerical results one of these polynomials has been conjectured [4, 13] to coincide with the classical limit ($q \to 1$) of the Drinfeld polynomial [17] which describes the irreducible representations of the loop algebra [11] in the sectors $2S^z = 0 \bmod N$. The previously formulated conjecture [1, 4, 13] that the regular XXZ Bethe vectors correspond to the highest weight vectors of the loop algebra has been investigated in [14] by means of the algebraic Bethe ansatz. Also here the results have been limited to the commensurate sectors $2S^z = 0 \bmod N$ where the algebraic structure of the symmetry generators has been identified. In [15, 16] the degeneracies of the periodic six-vertex model have been investigated from a different point of view by applying representation theory to construct analogues of Baxter's Q-operator. In [16] the classical Drinfeld polynomial has been identified in the spectrum of these Q-operators for several explicit examples when $N = 3$.

Clearly, the advantage of imposing the quasi-periodic boundary conditions $\lambda = q^{\pm 2S^z}$ is that the symmetry algebra is now known for all spin sectors while at the same time leaving the algebraic structure of the Bethe ansatz largely unchanged. This makes the twisted model (25) an ideal candidate for representation theoretic investigations and one can expect to find similar results as for the periodic case. Of particular interest in this context is also the study of finite-size effects in the thermodynamic limit, similar to those done in existing numerical investigations of the twisted XXZ chain e.g. [18–21]. These issues will be addressed in a forthcoming paper [23][3].

Finally, it needs to be pointed out that the proof of the infinite-dimensional symmetries given in this paper has only made use of the intertwining property of the monodromy matrix. This property is common to a large class of integrable vertex models associated with trigonometric solutions to the Yang–Baxter equation and quantum affine (super)algebras. Despite the obvious modifications in the algebraic structure of the Yang–Baxter algebra, we expect that the results found here can be extended to these models in a similar way as the periodic case has been generalized to other models in [8, 9] (albeit with different methods).

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